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On an extension of Holmgren's uniqueness theorem

by

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§1. **Introduction.** Consider a linear partial differential equation

$$D_t^m u + \sum_{j=1}^{m} \sum_{|\alpha| \leqq j} a_{j\alpha}(t, x) D_x^\alpha D_t^{m-j} u = 0 , \tag{1.1}$$

where $t \in \boldsymbol{R}$(=real numbers), $D_t = \partial/\partial t$, $x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D_x^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. $a_{j\alpha}$ are complex valued given functions and u is a complex valued unknown function. We assume that each $a_{j\alpha}$ is analytic in x, but not necessarily in t.

In an earlier paper [5] one of the present authors proved that the solution of the local Cauchy problem for (1.1) is unique. Thus he extended the well known theorem of Holmgren, in which the analyticity of $a_{j\alpha}$ both in x and in t was assumed.

Here we shall present another proof which is as direct as the original proof of Holmgren's uniqueness theorem. Thus we shall not use analytic functionals and the dual Cauchy-Kovalevskij theorem formulated for their Fourier-Borel transforms. Instead in §2 we give an existence and uniqueness theorem for the adjoint equation of (1.1) when the data are analytic. In our proof we use a variant of the method used earlier by Nagumo [3], Yamamoto [7] and Yamanaka [8] in different versions. We also give estimates of the solution by means of the data. Although we here treat Holmgren's uniqueness theorem we want to point out that the results in §2 themselves give a very sharp version of the Cauchy-Kovalevskij theorem for linear equations. They also make it possible to treat (1.1) directly without rewriting it as a system as was done in [5]. Theorem 2.1 has many other direct or indirect applications. Thus an argument of §4 type and Theorem 2.1 slightly reformulated gives another proof of the global version of the Cauchy-Kovalevskij theorem given in [4, Theorem 3].

In §3 the results of §2 together with the cut off method invented in [5] give a local uniqueness theorem. It turns out to be a little bit sharper than the corresponding theorem in [5] as far as classical solutions are concerned. However it is not hard to extend our result to cover distribution solutions, too. But such an extension would not be essential and we omit it.

When the coefficients are analytic in all variables one often gets lower bounds for the velocity of propagation of perturbations than for that of propagation of zeros. See J. Persson [6]. This depends on less straight forward algebraic properties of the principal polynomial than those used in defining our constant L in (2.7) below. See L. Hörmander [1, Cor. 5.3.3, p. 130], which is a generalization to distribution solutions of an idea by F. John [2]. This idea is used in a systematic way in [6]. The question now arises if the results in the present paper could be improved in a similar way also in cases when the coefficients are only continuous in t-variable. We have no answer to this question.

We do not repeat the argument of [5] in the case when the coefficients are analytic in all variables. See also [6].

Our local theorem gives a simple relation between the domain of dependence of the solution and the coefficients of the equation. In § 4 we use this relation to connect the local Cauchy problem with the global one in a very natural way. We shall state a global uniqueness theorem (Theorem 4.2) which slightly improves the corresponding theorem in [5] (Theorem 4').

We shall use the following notation.

For $x=(x_1, \cdots, x_n)\in \boldsymbol{C}^n$($\boldsymbol{C}$—complex numbers) we write (except for the case of a multi-index) $|x|=\max_{1\leqq i\leqq n}|x_i|$.

For $z_0\in \boldsymbol{C}^n$, $x_0\in \boldsymbol{R}^n$, $r>0$ we write

$$V_C(z_0, r)=\{z\in \boldsymbol{C}^n \mid |z-z_0|<r\} \quad \text{and} \quad V_R(x_0, r)=\{x\in \boldsymbol{R}^n \mid |x-x_0|<r\} .$$

For $A\subset \boldsymbol{C}^n$, $r>0$ we write $V_C(A, r)=\bigcup_{z\in A} V_C(z, r)$. For $B\in \boldsymbol{R}^n$ we define $V_R(B, r)$ similarly.

Let S be a subset of $\boldsymbol{R}$ and U be an open subset of $\boldsymbol{C}^n$. We denote by $CA(S, U)$ the set of all complex valued functions $f(t, z)$ which are continuous in (t, z) and holomorphic in z for $(t, z)\in S\times U$. Let S be a subset of $\boldsymbol{R}$, D be a subset of $\boldsymbol{R}^n$ and U be an open subset of $\boldsymbol{C}^n$ such that $D\subset U$. A complex valued function $f(t, x)$ defined for $(t, x)\in S\times D$ is said to belong to $CA(S, D, U)$, if it can be extended to a function $f(t, z)$ which is in $CA(S, U)$.

Let A be a subset of $\boldsymbol{C}^n$ or of $\boldsymbol{R}^n$. For a complex valued bounded function $f(z)$ defined in A we write $\|f\|_A=\sup|f(z)|$.

§ 2. Existence of solutions for the adjoint equation. By the adjoint equation of (1.1) we mean here the equation

$$D_t^m v+\sum_{j=1}^m \sum_{|\alpha|\leqq j}(-1)^{j+|\alpha|}D_t^{m-j}D_z^\alpha(a_{j\alpha}(t, z)v)=0 , \tag{2.1}$$

where $z=(z_1, \cdots, z_n)\in \boldsymbol{C}^n$ and the differentiation D_z^α is the complex differentiation $\partial^{|\alpha|}/\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}$. We want to construct a solution $v(t, z)$ of (2.1) which satisfies the initial condition

$$D_t^{m-1}v(t_0, z)=\varphi(z), \quad D_t^{m-j}v(t_0, z)=0 \quad (j=2, \cdots, m) , \tag{2.2}$$

where φ is a given holomorphic function of z.

If $a_{j\alpha}$ is sufficiently smooth with respect to t, then the Cauchy problem (2.1)-(2.2) is equivalent to a single integro-differential equation

$$v(t, z)+P(v, t, z)=\psi(t, z) , \tag{2.3}$$

where

$$P(v, t, z)=\sum_{j=1}^{m}\int_{t_0}^{t}\left[\frac{(\tau-t)^{j-1}}{(j-1)!}\sum_{|\alpha|\leqq j}(-1)^{|\alpha|}D_z^{\alpha}(a_{j\alpha}(\tau, z)v(\tau, z))\right]d\tau \tag{2.4}$$

and

$$\psi(t, z)=\frac{(t-t_0)^{m-1}}{(m-1)!}\varphi(z) . \tag{2.5}$$

If $a_{j\alpha}$ are not sufficiently smooth, then a solution of (2.3) is not necessarily a solution of the Cauchy problem. We do not, however, want to assume the smoothness of $a_{j\alpha}$ and it is in fact sufficient for our purpose to solve the equation (2.3). So we shall do that.

Before stating the existence theorem it is convenient for us to state the following

LEMMA. *Let A and B be subsets of C^n such that $V_C(A, r)\subset B$, where $r>0$. Let $f(z)$ be a complex valued function on B which is bounded and holomorphic in the interior of B. Then, for any multi-index α, the inequality*

$$\| D_z^{\alpha}f \|_A=\frac{\alpha!}{r^{|\alpha|}}\| f \|_B \tag{2.6}$$

holds. (As usual we write $\alpha!=\alpha_1! \cdots \alpha_n!$.)

The proof is immediate by the Cauchy formula.

THEOREM 2.1. *Let c be a given point of C^n and let M and T be given positive constants. In the equation (2.1) assume that each coefficient $a_{j\alpha}(t, z)$ is a bounded function belonging to $CA([0, T], V_C(c, M))$. In the initial condition (2.2) assume that $0\leqq t_0\leqq T$ and φ is holomorphic in z and bounded on $V_C(c, M)$. Put*

$$A_{j\alpha}=\sup_{0\leqq t\leqq T}\| a_{j\alpha}\|_{V_C(c,M)} , \qquad L=m\cdot e\cdot \max_{1\leqq j\leqq m}\Big(\sum_{|\alpha|=j}\alpha!\, A_{j\alpha}\Big)^{1/j} . \tag{2.7}$$

Then there exists a unique solution $v(t, z)$ of the equation (2.3) which is defined for (t, z) such that

$$0\leqq t\leqq t_0 , \qquad |z-c|<M-L(t_0-t) , \tag{2.8}$$

and which is continuous in (t, z) and holomorphic in z.

Proof. We solve the equation (2.3) by successive approximation. Thus we define the approximations $v_p(t, z)$ $(p=0, 1, 2, \cdots)$ of $v(t, z)$ by

$$v_0(t, z)=\psi(t, z), \quad v_p(t, z)=-P(v_{p-1}, t, z)+\psi(t, z) \quad (p\geqq 1)\,. \tag{2.9}$$

We shall see that v_p converges as $p\to\infty$ to a solution v of (2.3). To do this we adopt here the following simplified norm notation:

$$\|w\|_r=\|w\|_{V_C(c,M-r)}=\sup\{|w(z)|\mid z\in V_C(c, M-r)\}\,, \tag{2.10}$$

where r is a non-negative number $<M$.

We assume first that $L>0$ and we put

$$K=1+\max_{1\leqq j\leqq m}\left[\frac{(\mathrm{me})^j}{L^j}\sum_{|\alpha|<j}\alpha!\,A_{j\alpha}\right].$$

In the inequalities below we use the simple fact that $K^j\geqq K$ and $h^{-|\alpha|}\leqq h^{-(j-1)}+1$, $h>0$, $|\alpha|\leqq j-1$, $j=1, 2, \cdots$. Let $w(z)$ be holomorphic in $V_C(c, M)$. Then, if $0\leqq r-h<r<M$, the lemma gives

$$\begin{aligned}\Big\|\sum_{|\alpha|\leqq j}D_z^\alpha(a_{j\alpha}w)\Big\|_r&\leqq\|w\|_{r-h}\Big(\sum_{|\alpha|=j}+\sum_{|\alpha|<j}\Big)\Big(\frac{\alpha!}{h^{|\alpha|}}A_{j\alpha}\Big)\\&\leqq\|w\|_{r-h}\Big(\frac{L}{me}\Big)^j\Big\{\frac{1}{h^j}+\Big(\frac{1}{h^{j-1}}+1\Big)K\Big\}\leqq\|w\|_{r-h}\Big(\frac{L}{me}\Big)^j\Big(\frac{1}{h}+K\Big)^j\,.\end{aligned} \tag{2.11}$$

Hence we have, if $0<r<M$ and $0\leqq t\leqq t_0$,

$$\begin{aligned}\|v_1(t,\cdot)-v_0(t,\cdot)\|_r&=\|P(\psi, t,\cdot)\|_r\\&\leqq\sum_{j=1}^m\Big(\frac{L}{me}\Big)^j\Big(\frac{1}{r}+K\Big)^j\int_t^{t_0}\frac{(\tau-t)^{j-1}}{(j-1)!}\|\psi(\tau,\cdot))\|_0\,d\tau\\&\leqq C_T\|\varphi\|_0\sum_{j=1}^m\Big(\frac{L}{me}\Big)^j\Big(\frac{1}{r}+K\Big)^j\frac{(t_0-t)^j}{j!}\,,\end{aligned} \tag{2.12}$$

where $C_T=T^{m-1}/(m-1)!$. Further we can prove for $p=1, 2, \cdots$ that the inequality

$$\|v_p(t,\cdot)-v_{p-1}(t,\cdot)\|_r\leqq C_T\|\varphi\|_0 m^{p-1}\sum_{j=p}^{mp}\Big(\frac{L}{me}\Big)^j\Big(\frac{p}{r}+K\Big)^j\frac{(t_0-t)^j}{j!} \tag{2.13}$$

holds, if $0<r<M$ and $0\leqq t\leqq t_0$. We prove (2.13) by induction. Indeed, (2.13) reduces to (2.12) for $p=1$. We assume that (2.13) is true for some p. Then we have by (2.11)

$$\begin{aligned}&\|v_{p+1}(t,\cdot)-v_p(t,\cdot)\|_r=\|P(v_p-v_{p-1}, t,\cdot)\|_r\\&\leqq\sum_{j=1}^m\Big(\frac{L}{me}\Big)^j\Big(\frac{p+1}{r}+K\Big)^j\int_t^{t_0}\frac{(\tau-t)^{j-1}}{(j-1)!}\|v_p(\tau,\cdot)-v_{p-1}(\tau,\cdot)\|_{pr/(p+1)}d\tau\\&\leqq C_T\|\varphi\|_0\,m^{p-1}\sum_{j=1}^m\sum_{k=p}^{mp}\Big(\frac{L}{me}\Big)^{j+k}\Big(\frac{p+1}{r}+K\Big)^{j+k}\int_t^{t_0}\frac{(\tau-t)^{j-1}(t_0-\tau)^k}{(j-1)!\,k!}d\tau\\&=C_T\|\varphi\|_0\,m^{p-1}\sum_{j=1}^m\sum_{k=p}^{mp}\Big(\frac{L}{me}\Big)^{j+k}\Big(\frac{p+1}{r}+K\Big)^{j+k}\frac{(t_0-t)^{j+k}}{(j+k)!}\\&\leqq C_T\|\varphi\|_0\,m^p\sum_{j=p+1}^{m(p+1)}\Big(\frac{L}{me}\Big)^j\Big(\frac{p+1}{r}+K\Big)^j\frac{(t_0-t)^j}{j!}\,,\end{aligned}$$

which shows that (2.13) is true for $p+1$. Thus we have proved that for all r, $0<r<M$, (2.13) is true for all p.

We shall now use the following three facts. We have

$$\left(1+\frac{Kr}{p}\right)^{mp}<e^{mKr},\ p=1,2,\cdots. \tag{2.14.1}$$

There exists a constant κ independent of p such that

$$p^j/j!\leqq\kappa e^j\ ,\quad j=p\ ,p+1,\cdots. \tag{2.14.2}$$

If $L(t_0-t)/r\leqq 1$, then we have

$$\sum_{j=p}^{mp}\left(\frac{L(t_0-t)}{r}\right)^j\leqq mp\left(\frac{L(t_0-t)}{r}\right)^p,\qquad p=1,2,\cdots. \tag{2.14.3}$$

Assume that $L(t_0-t)/r\leqq 1$. From (2.14.1)-(2.14.3) and the inequality (2.13) we now get

$$\begin{aligned}
&\|v_p(t,\cdot)-v_{p-1}(t,\cdot)\|_r\\
&\leqq C_T\|\varphi\|_0\, m^{-1}\left(1+\frac{Kr}{p}\right)^{mp}\sum_{j=p}^{mp}\left(\frac{L(t_0-t)}{er}\right)^j\frac{p^j}{j!}\\
&\leqq C_T\|\varphi\|_0\, m^{-1}e^{mKr}\sum_{j=p}^{mp}\left(\frac{L(t_0-t)}{r}\right)^j\\
&\leqq C_T\kappa\|\varphi\|_0\, e^{mKr}p\left(\frac{L(t_0-t)}{r}\right)^p.
\end{aligned} \tag{2.15}$$

For fixed r it follows from (2.15) that the series $\sum_{p=1}^{\infty}\|v_p(t,\cdot)-v_{p-1}(t,\cdot)\|_r$ converges uniformly on all compact subset of $\{t\mid L(t_0-t)/r<1,\ 0\leqq t\leqq t_0\}$. Hence we conclude that the limit

$$v(t,z)=\lim_{p\to\infty}v_p(t,z)=\psi(t,z)+\sum_{p=1}^{\infty}(v_p(t,z)-v_{p-1}(t,z)) \tag{2.16}$$

exists for $t_0-t<r/L$ and $z\in V_C(c, M-r)$ and solves the equation (2.3). In other words[1], the limit $v(t,z)$ exists for (t,z) satisfying (2.8) and solves the equation (2.3).

In the above argument we have assumed that $L>0$. In case of $L=0$ we can argue as follows. We first replace $L(=0)$ by a small positive parameter λ. It is then proved that the solution exists for (t,z) such that $0\leqq t\leqq t_0$, $|z-c|<M-\lambda(t_0-t)$. We then let $\lambda\to 0$ and we know that the solution exists for (t,z) satisfying (2.8) with $L=0$.

The proof of the uniqueness is easy by the standard method of constructing an estimate similar to (2.12) for the difference of two solutions and is omitted. *q.e.d.*

We have thus proved the unique existence of the solution of (2.3). For our purpose, however, we need to know more. We need to know

1) We eliminate the parameter r from the two relations $t_0-t<r/L$ and $z\in V_C(c, M-r)$.

how the solution depends on the initial data. The following theorem will meet our requirement.

THEOREM 2.2. *Let $v(t, z)$ be the unique solution of the equation* (2.3) *whose existence is established by Theorem* 2.1. *Let t be any number such that $0 \leqq t \leqq t_0$ and $t_0 - t < M/L$. Let θ be any number such that $1 < \theta < M(L(t_0-t))^{-1}$. Let A be any set $\subset V_C(c, M-\theta L(t_0-t))$. Then there exists a constant C_θ depending on θ, but not on the initial function φ such that*

$$\| v(t, \cdot) \|_A \leqq C_\theta \| \varphi \|_{V_C(A, \theta L(t_0-t))} . \tag{2.17}$$

Proof. Define by (2.9) the successive approximations $v_p(t, z)$ ($p = 0, 1, \cdots$) of $v(t, z)$. We have (2.16). Hence

$$\| v(t, \cdot) \|_A \leqq \| \psi(t, \cdot) \|_A + \sum_{p=1}^{\infty} \| v_p(t, \cdot) - v_{p-1}(t, \cdot) \|_A .$$

We note that $V_C(A, \theta L(t_0-t)) \subset V_C(c, M)$. Let w be holomorphic in $V_C(c, M)$. For δ in $0 \leqq \delta \leqq \theta L(t_0-t)$ define

$$| w |_\delta = \| w \|_{V_C(A, \theta L(t_0-t)-\delta)} .$$

We use the lemma as we did when we derived (2.15). If $0 < t' \leqq t_0$ and $L(t_0-t')/\delta < 1$, then we get

$$| v_p(t', \cdot) - v_{p-1}(t', \cdot) |_\delta \leqq C_T \kappa \, | \varphi |_0 \, e^{Km\delta} p \left(\frac{L(t_0-t')}{\delta} \right)^p .$$

We let $\delta = \theta L(t_0-t)$ and $t = t'$. We note that then $L(t_0-t')/\delta = \theta^{-1} < 1$ and $\| \cdot \|_A = | \cdot |_\delta$. So we get

$$\| v_p(t, \cdot) - v_{p-1}(t, \cdot) \|_A$$
$$\leqq C_T \kappa \, \| \varphi \|_{V_C(A, \theta L(t_0-t))} \, e^{mK\theta L(t_0-t)} \frac{1}{\theta^p}$$

and

$$\| \psi(t, \cdot) \|_A \leqq C_T \| \varphi \|_A \leqq C_T \| \varphi \|_{V_C(A, \theta L(t_0-t))} .$$

Hence we can put

$$C_\theta = C_T \left(1 + \kappa e^{mK\theta L t_0} \sum_{p=1}^{\infty} \frac{1}{\theta^p} \right) \tag{2.18}$$

and we have (2.17). ***q.e.d.***

§ 3. **Local uniqueness of non-analytic solutions.** In this section we fix a point c of $\boldsymbol{R}^n$ and two positive constants M and T and we assume throughout that the coefficients $a_{j\alpha}(t, x)$ of the equation (1.1) belong to $CA([0, T], V_R(c, M), V_C(c, M))$ and that the extended functions $a_{j\alpha}(t, z)$ are bounded on $[0, T] \times V_C(c, M)$. We define the constant L by (2.7). The purpose of this section is to prove the following

THEOREM 3.1. *Let $u(t, x)$ be a solution of the equation* (1.1) *which is defined and m times continuously differentiable for*

$$0 \leqq t \leqq T \quad \text{and} \quad x \in V_R(c, M-Lt) \tag{3.1}$$

and which satisfies the initial condition

$$x \in V_R(c, M) \Longrightarrow D_t^j u(0, x) = 0 \qquad (j=0, 1, \cdots, m-1)\,. \tag{3.2}$$

Then $u(t, x)$ vanishes where it is defined.[1]

The proof of the theorem is performed in three steps. We shall state the first two steps as lemmas.

LEMMA 3.1. *Let r be any number such that $0<r<M$. Let $u(t, x)$ be a solution of* (1.1) *which is defined and m times continuously differentiable for* (3.1) *and which satisfies the initial condition*

$$x \in V_R(c, r) \Longrightarrow D_t^j u(0, x) = 0 \qquad (j=0, 1, \cdots, m-1)\,.$$

Then $u(t, x)$ vanishes for (t, x) such that

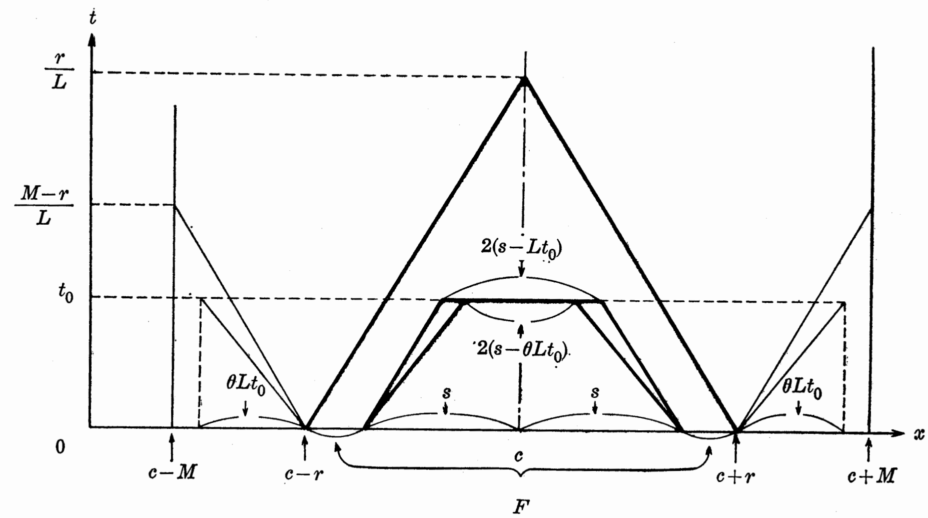
$$0 \leqq t \leqq \min(T, (M-r)/L)\,, \qquad x \in V_R(c, r-Lt)\,.$$

Proof. Take arbitrarily a fixed number t_0 such that

$$0<t_0<\min(T, (M-r)/L) \quad \text{and} \quad r-Lt_0>0\,.$$

Then take an arbitrary s such that $0<s<r$ and $s-Lt_0>0$. Let $\rho(x) \in C^\infty(\boldsymbol{R}^n)$ such that supp. $(\rho) \subset V_R(c, r)$ and supp. $(\rho-1) \subset \{x \in \boldsymbol{R}^n \mid |x-c|>s\}$.
We put $w(t, x)=\rho(x)u(t, x)$ and define $f(t, x)$ by

$$D_t^m w + \sum_{j=1}^{m} \sum_{|\alpha| \leqq j} a_{j\alpha}(t, x) D_x^\alpha D_t^{m-j} w = f(t, x)\,.$$



[1] See the Remark at the end of this paragraph.

Thus $\rho(x)$ is what was introduced in [5] as a cut-off function. The function w satisfies the same initial condition as u and supp. $(w(t, \cdot))\subset V_C(c, r)$. Note also that

$$\text{supp.}\ (f(t, \cdot))\subset F=\{x\in \boldsymbol{R}^n \mid s<|x-c|<r\}\ .$$

Let $\varphi(z)$ be any bounded holomorphic function on $V_C(c, M)$ and $v(t, z)$ be the holomorphic solution of the equation (2.3) with $\psi(z)$ defined by (2.5). Put

$$I(\varphi, t_0)=\int_{V_{\boldsymbol{R}}(c,r)} w(t_0, x)\varphi(x)dx\ . \tag{3.3}$$

We use (3.2) and partial integration in t. We get

$$\begin{aligned} I(\varphi, t_0) &=(-1)^m \int_{V_C(c,r)} \varphi(x)dx \int_0^t D_t^m w(t, x)\frac{(t-t_0)^{m-1}}{(m-1)!}dt \\ &=(-1)^m \int_0^t dt \int_{V_C(c,r)} D_t^m w(t, x)\psi(t, x)dx \\ &=(-1)^m \int_0^{t_0} dt \int_{V_{\boldsymbol{R}}(c,r)} D_t^m w(t, x)(v(t, x)+P(v, t, x))dx\ . \end{aligned}$$

We use (3.2), (2.4) and the fact that w has compact support in $V_R(c, r)$ for fixed t when we continue the partial integration. At last the definition of $f(t, x)$ gives

$$\begin{aligned} I(\varphi, t_0) &=(-1)^m \int_0^{t_0} dt \int_{V_{\boldsymbol{R}}(c,r)} f(t, x)v(t, x)dx \\ &=(-1)^m \int_0^{t_0} dt \int_F f(t, x)v(t, x)dx\ . \end{aligned}$$

Hence, if we put

$$C_f=t_0\cdot \sup_{0\leqq t\leqq t_0} \|f(t, \cdot)\|_F\cdot \int_F dx\ ,$$

we have

$$|I(\varphi, t_0)|\leqq C_f\cdot \sup_{0\leqq t\leqq t_0} \|v(t, \cdot)\|_F\ . \tag{3.4}$$

We can further estimate the right hand side of (3.4) by Theorem 2.2. Thus, let θ be any number such that

$$\theta>1, \qquad M-\theta Lt_0>r \quad\text{and}\quad s-\theta Lt_0>0\ .$$

(Since $M-Lt_0>r$ and $s-Lt_0>0$, we can certainly choose such a θ.) We note that $F\subset V_C(c, r)$ so $V_C(F, \theta Lt_0)\subset V_C(c, r+\theta Lt_0)\subset V_C(c, M)$. Then we have by Theorem 2.2

$$\sup_{0\leqq t\leqq t_0} \|v(t, \cdot)\|_F\leqq C_\theta\ \|\varphi\|_{V_C(F,\theta Lt_0)}\ . \tag{3.5}$$

Now (3.4) and (3.5) yields

$$|I(\varphi, t_0)| \leqq C_f C_\theta \|\varphi\|_{V_C(F, \theta L t_0)} . \tag{3.6}$$

Take an arbitrary C^∞ function $h(x)$ on $\boldsymbol{R}^n$ such that supp. $(h) \subset V_R(c, s-\theta L t_0)$. For $k=1, 2, \cdots$ we put

$$\varphi_k(z)=\left(\frac{k}{\sqrt{\pi}}\right)^n \int_{R^n} h(x) \exp\left(-k^2 \sum_{i=1}^n (z_i-x_i)^2\right) dx .$$

Each φ_k is an entire function on $\boldsymbol{C}^n$ and it is easily seen that as $k \to \infty$

$$\varphi_k(x) \text{ converges to } h(x) \text{ uniformly on } \boldsymbol{R}^n \tag{3.7}$$

and

$$\varphi_k(z) \text{ converges to } 0 \text{ uniformly on } \boldsymbol{C}^n - V_C(c, s-\theta L t_0) . \tag{3.8}$$

Since $V_C(F, \theta L t_0) \subset \boldsymbol{C}^n - V_C(c, s-\theta L t_0)$, (3.6) and (3.8) yields

$$\lim_{k \to \infty} I(\varphi_k, t_0)=0 . \tag{3.9}$$

On the other hand (3.3) and (3.7) yields

$$\lim_{k \to \infty} I(\varphi_k, t_0)=\int_{V_R(c, r)} w(t_0, x) h(x) dx . \tag{3.10}$$

By (3.9) and (3.10) we obtain

$$\int_{V_R(c, r)} w(t_0, x) h(x) dx=0 . \tag{3.11}$$

Since $h(x)$ is an arbitrary C^∞ function such that supp. $(h) \subset V_R(c, s-\theta L t_0)$, we conclude from (3.11) that

$$x \in V_R(c, s-\theta L t_0) \quad \text{implies} \quad u(t_0, x)=w(t_0, x)=0 . \tag{3.12}$$

Since s can be arbitrarily close to r and θ can be as close to 1 as one likes, we conclude from (3.12) that

$$x \in V_R(c, r-L t_0) \quad \text{implies} \quad u(t_0, x)=0 ,$$

which completes the proof of the lemma.

Using the above lemma repeatedly in the t direction, we easily obtain the following improvement of Lemma 3.1.

LEMMA 3.2. *Under the same assumptions as in Lemma* 3.1, $u(t, x)$ *vanishes for* (t, x) *such that*

$$0 \leqq t \leqq T , \qquad x \in V_R(c, r-L t) .$$

Finally we note that the number r in Lemma 3.2 can be taken as close to M as we like. Then we immediately obtain Theorem 3.1.

Remark. If one reads the following paragraph, he will realize that the assumptions in Theorem 3.1 can be relaxed as follows:

1) Instead of the condition

$$a_{j\alpha} \in CA([0, T], V_R(c, M), V_C(c, M))$$

it is enough to assume that

$$a_{j\alpha}\in CA([0,\ T],\ V_R(c,\ M),\ \Omega)\ ,$$

where Ω is an arbitrary open neighbourhood of $V_R(c, M)$ in C^n.

2) The constant L defined by (2.7) can be replaced by

$$L'=m\cdot e\cdot \max_{1\leqq j\leqq m}\ (\sum_{|\alpha|=j}\alpha!\ A'_{j\alpha})^{1/j}\ ,\quad \text{where}\quad A'_{j\alpha}=\sup_{0\leqq t\leqq T}\|a_{j\alpha}\|_{V_R(c,M)}\ .$$

In other words, our bound of the propagation speed is determined by the absolute values of the coefficients *in the real domain.*

§4. Global uniqueness of solutions. In this section we consider the Cauchy problem whose Cauchy data are given on the whole space R^n. In order to establish the uniqueness of the solution of such global problems it suffices to superpose with respect both to x and to t the local uniqueness such as asserted by Theorem 3.1. In this manner we can prove the following

THEOREM 4.1. *Let Ω be an open set in C^n such that $R^n\subset\Omega$ and let $T>0$ be a constant. In* (1.1) *assume that each coefficient $a_{j\alpha}(t, x)$ belongs to $CA([0, T], R^n, \Omega)$. Let*

$$L(t,\ x)=m\cdot e\cdot \max_{1\leqq j\leqq m}\ (\sum_{|\alpha|=j}|\ a_{j\alpha}(t,\ x)\ |\ \alpha!)^{1/j}$$

and let

$$\lambda(r)=\max(1\ ,\ \sup_{|x|\leqq r,0\leqq t\leqq T}L(t,\ x))\ ,\quad r\geqq 0\ .$$

Assume that

$$\int_0^\infty\frac{1}{\lambda(r)}dr=\infty\ . \tag{4.1}$$

Let $u(t, x)$ be a solution of (1.1) *which is defined and m times continuously differentiable in* $[0, T]\times R^n$ *and satisfies the initial condition*

$$D_t^j u(0,\ x)=0\ ,\quad x\in R^n\ ,\quad j=0,\ 1,\ \cdots,\ m-1\ . \tag{4.2}$$

Then $u(t, x)$ vanishes where it is defined.

Proof. Let $M>0$, $0\leqq t'<T$, $x'\in R^n$, $\varepsilon>0$ be constants. We assume that $t'+\varepsilon\leqq T$. Define

$$A_{j\alpha}=\sup_{t'\leqq t\leqq t'+\varepsilon}\|a_{j\alpha}(t,\ z)\|_{V_C(x',M)}\ .$$

We assume that M is so small that $A_{j\alpha}$ is well defined. Let

$$L=m\cdot e\cdot \max_{1\leqq j\leqq m}\ (\sum_{|\alpha|=j}A_{j\alpha}\alpha!)^{1/j}\ .$$

If the Cauchy data of u for $t=t'$ and $|x-x'|<M$ are zero, then a translated version of Theorem 3.1 says that $u(t, x)$ is zero in that part of

the cone $|x-x'|<M-L(t-t')$ where it is defined. If we let $M\to 0$ and $\varepsilon\to 0$ at the same time, then $L\to L(t', x')$. In view of Theorem 3.1 $L(t', x')$ is a local bound of the possible velocities of propagation of perturbations in the Cauchy data. That means the following. For any $\theta>1$ there is a neighbourhood U of (t', x') for which the velocity is less than $\theta L(t', x')$. Let $t_0\in[0, T]$ and $x^0\in \boldsymbol{R}^n$. We shall prove that $u(t_0, x^0)=0$. We shall do this somewhat loosely, but it is clear from below how it could be done in a strict way. Let $r^0=|x^0|$. From the argument above follows that $\lambda(r)$ is a bound of the velocity of propagation of perturbations in the Cauchy data in $[0, T]\times\{x \mid |x|\leqq r\}$. Let $R(t)$ be defined by

$$-\int_{r^0}^{R(t)}\frac{1}{\lambda(r)}dr=t_0-t\ , \qquad t<t_0\ .$$

We note that $(dR/dt)=\lambda(R(t))$. $R(t)$ is well defined by (4.1). It is now clear that $u(x, t_0)=0$ if $|x|\leqq r^0$, since the Cauchy data of u at $t=0$ is zero for all x and especially for $|x|\leqq R(0)$. The proof is completed.

Note that the above theorem is very similar to Theorem 4′ of [5], but our present theorem is slightly wider than the former one with respect to the restriction on the growth of the coefficients at the infinity. Thus, for instance, the condition

$$|\alpha|=j \Longrightarrow a_{j\alpha}(t, x)=O((|x|\cdot\log|x|)^j) \tag{4.3}$$

is sufficient in order to use the above Theorem 4.1, while a condition such as

$$|\alpha|=j \Longrightarrow a_{j\alpha}(t, x)=O(|x|^j\cdot\log|x|)\ ,$$

which is more restrictive than (4.3), will be required, if Theorem 4′ of [5] is used.

Counter examples of non-uniqueness in the case when the condition (4.1) is not satisfied can easily be constructed. For example, let $\lambda(r)>0$ and $f(r)\not\equiv 0$ be any function $\in C^\infty(\boldsymbol{R})$. Then, for any $m=1, 2, \cdots$, the function

$$u(t, x)=f\left(t+\int_0^x\frac{dr}{\lambda(r)}\right)$$

is a solution of a differential equation of the form

$$\frac{\partial^m u}{\partial t^m}=(\lambda(x))^m\frac{\partial^m u}{\partial x^m}+\sum_{j=1}^{m-1}a_j(x)\frac{\partial^j u}{\partial t^j}\ .$$

Assume that

$$\int_0^\infty\frac{dr}{\lambda(r)}=c<\infty$$

and let $f(r)$ be such that supp. $(f)\subset\{r \mid r\geqq c\}$. Then u satisfies the initial condition (4.2), but it is not identically zero.

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